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A Hardware-efficient Training-oriented Quadratic Approximation Softmax for Transformers

ARITH 2026

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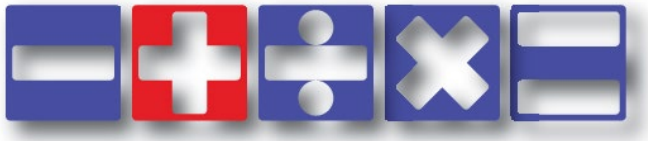

Fulda, Germany. June 28 - July 1, 2026.

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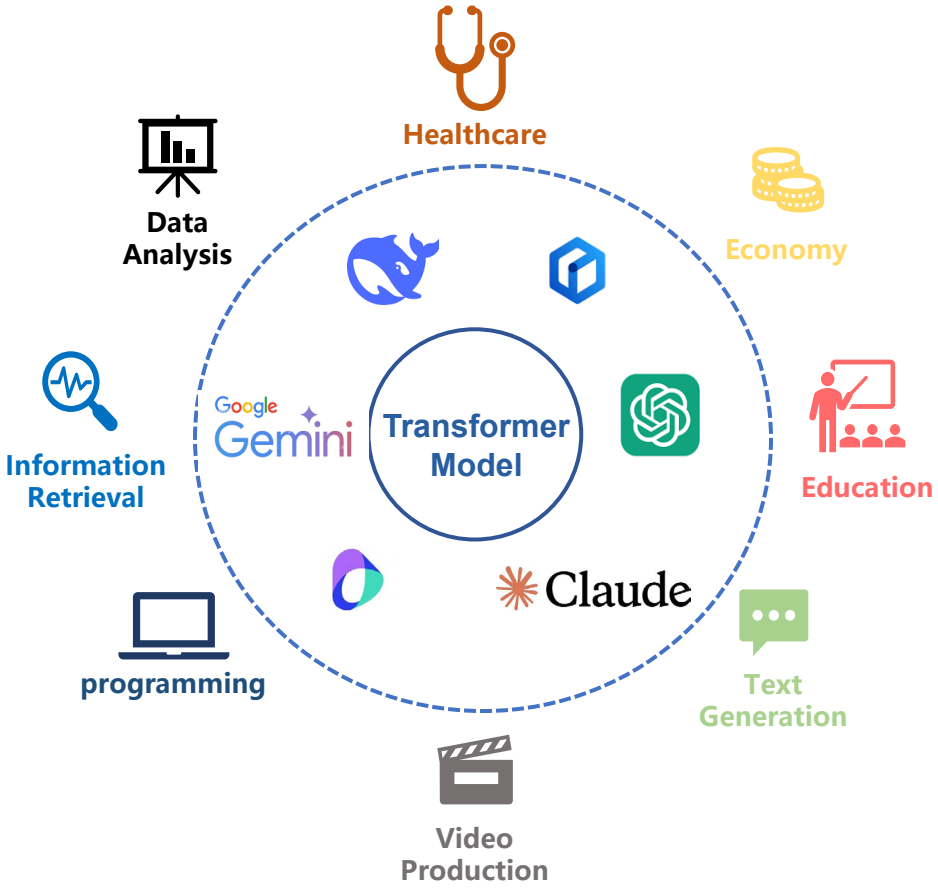
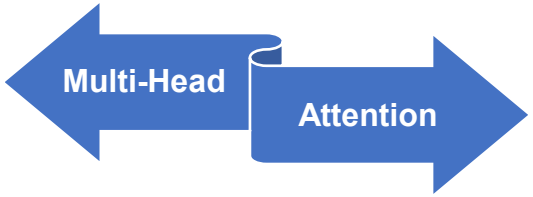
Content

Background: Softmax in Transformer Architectures

Softmax is the core computational component of the self-attention mechanism in Transformer networks.





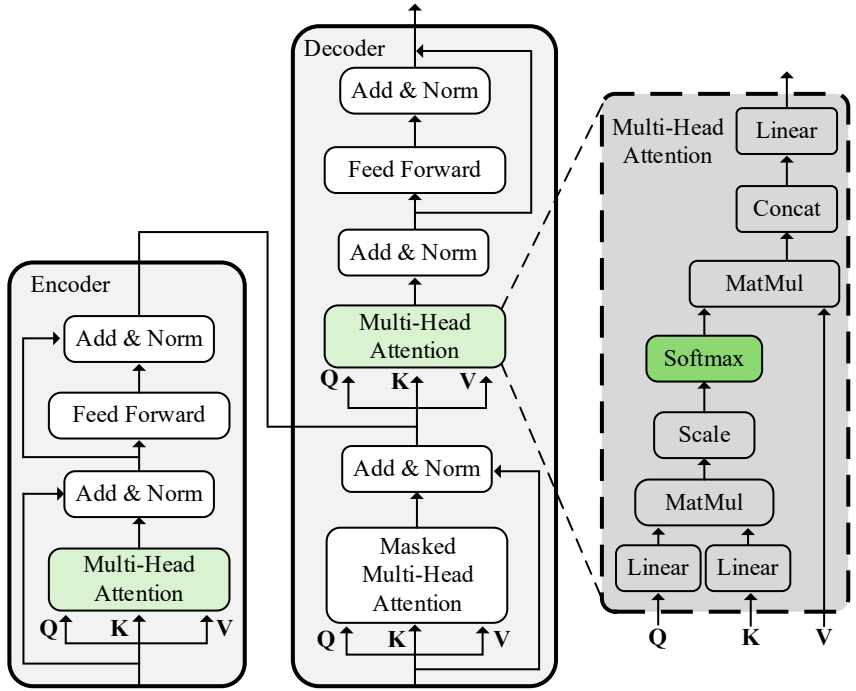


Fig.1 : Multi-Head Attention mechanism and the role of Softmax in Transformer blocks

$$softmax(x_i) = \frac{e^{x_i}}{\sum e^{x_i}}$$

Problem Statement: The Computational Bottleneck

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Softmax has emerged as a primary computational bottleneck in Transformer-based network.

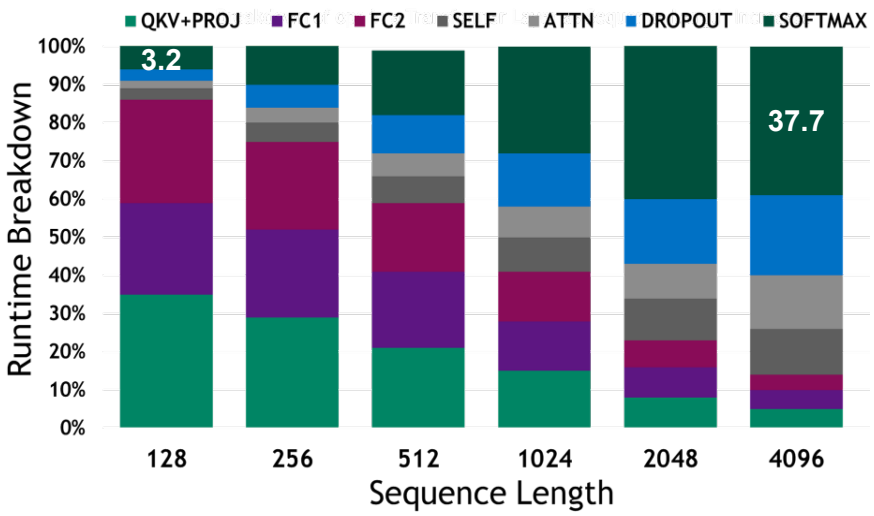
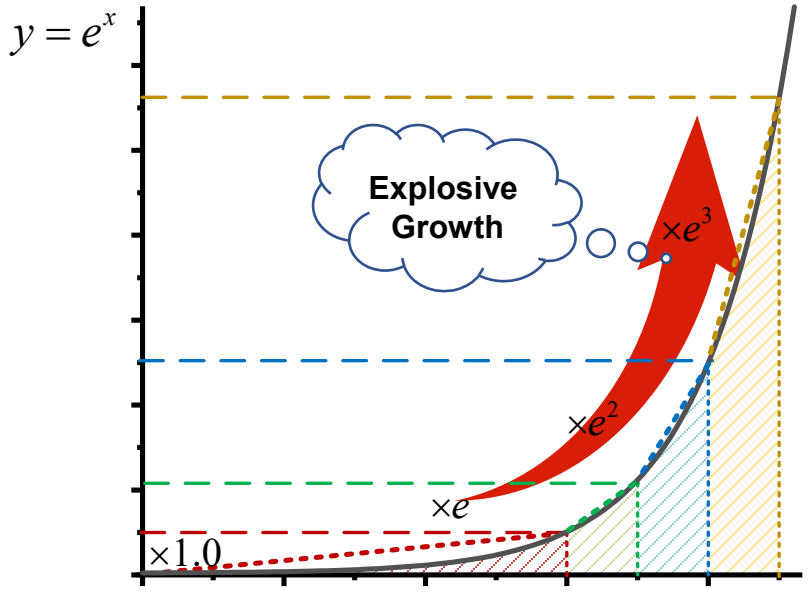
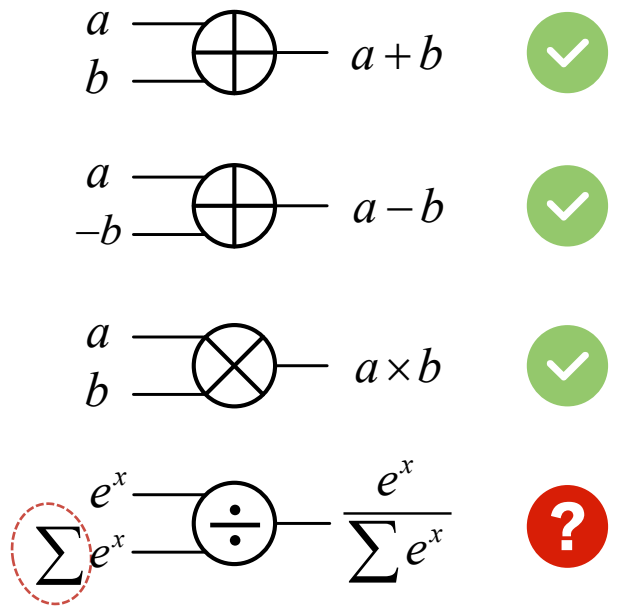


Fig. 2: Runtime breakdown for BERT-Large on a Volta GPU[10].



Growth of exponent & Nonlinearity



Global Reduction & expensive divisor operator

[10] J. R. Stevens, R. Venkatesan, S. Dai, B. Khailany, and A. Raghunathan, "Softermax: Hardware/software co-design of an efficient softmax for transformers," in 2021 58th ACM/IEEE Design Automation Conference (DAC). IEEE, 2021, pp. 469–474.

Motivation: Training-oriented High-Precision Requirements

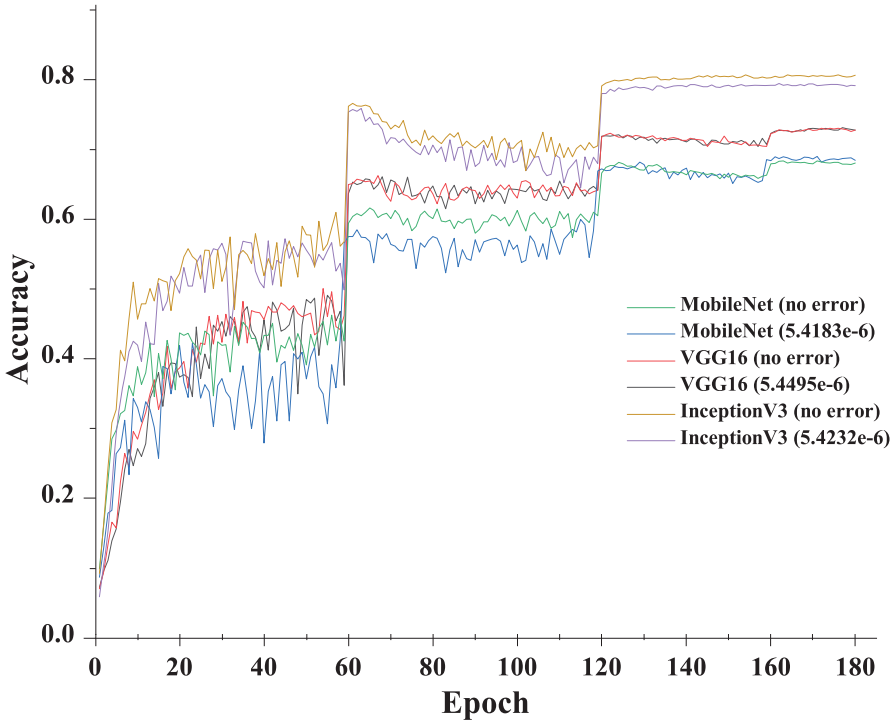


Fig3. Training accuracy of different networks with exact and approximate softmax[12].

High-Precision Requirement

Lookup Table : Excessive Storage

PiecewiseLinear : Too Many Segments

C O R D I C : Long Iteration Cycles

Error Analysis

More Analytic Approximation

Contributions:

- **TQA Error analysis**
- **TQA Softmax Hardware architecture**
- **Synthesis and Application Evaluation**

[12] Y. Zhang, L. Peng, L. Quan, Y. Zhang, S. Zheng, and H. Chen, “High precision method and architecture for base-2 softmax function in DNN training,” IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 70, no. 8, pp. 3268–3279, 2023.

Content

Base-2 Softmax and Down-scaling Strategy

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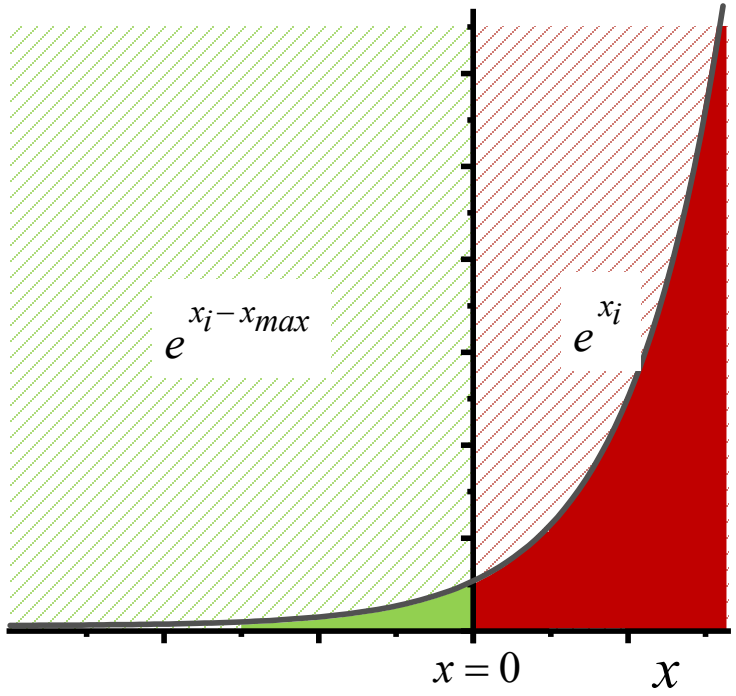
From a hardware perspective, Base-2 Softmax is a more suitable choice.



$$e^{m} = e^{m_1+m_2} = e^{m_1} \cdot e^{m_2} = e^{m_1} \gg / \ll m_2$$

$$2^{n} = 2^{n_1+n_2} = 2^{n_1} \cdot 2^{n_2} = 2^{n_1} \gg / \ll n_2$$

$e^m = 2^n$

$(n = m \times \log_2 e)$

$$softmax(x_i) = \frac{e^{x_i}}{\sum e^{x_i}} = \frac{e^{x_i} / e^{x_{max}}}{\sum e^{x_i} / e^{x_{max}}} = \frac{e^{x_i - x_{max}}}{\sum e^{x_i - x_{max}}}$$

Hardware Implementation

$$base-2\ softmax(x_i) = \frac{2^{x_i}}{\sum 2^{x_i}} = \frac{2^{x_i} / 2^{x_{max}}}{\sum 2^{x_i} / 2^{x_{max}}} = \frac{2^{x_i - x_{max}}}{\sum 2^{x_i - x_{max}}}$$

Log-Sum-Exp(LSE) and Computational Decomposition

LSE skill and computational decomposition are leveraged to optimize hardware design.

Log-Sum-Exp skill

- Decompose Softmax into exponent and log domains using LSE skill.

$$\log_2(f_2(x)) = x_i^{'} - \log_2(\sum 2^{x_j^{'}})$$

$$(x_i^{'} = x_i - x_{max})$$

- Division operator is substituted by **subtraction** in the log domain

- Transform softmax into a sequence of **two exponent** and **one logarithm** operations for optimization

$$f_2(x_i) = \frac{A}{B}$$

LSE skill

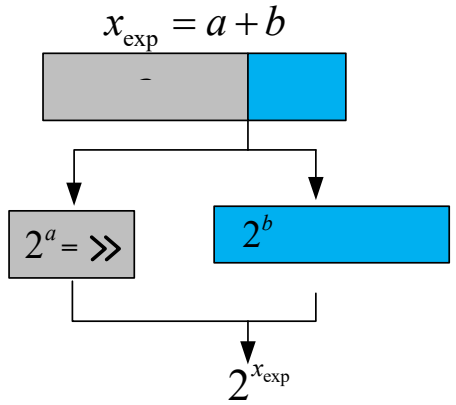
$$f_2(x_i) = 2^{A^{'} - B^{'}}$$

Exponent Decomposition

- Decompose input into
  a **negative integer a**
  and
  a **positive fraction b.**

$$x_{\exp} = a + b,\ a \in \mathbb{Z}^{-}, b \in [0,1)$$

$$2^{x_{\exp}} = 2^{b} \gg (-a)$$

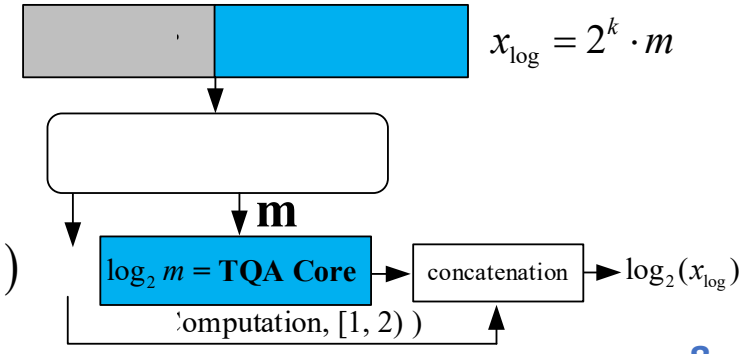


Logarithm Decomposition

- Decompose input into
  a **integer part k**
  and
  a **mantissa part b.**

$$x_{\log} = 2^{k} \cdot m,\ k \in \mathbb{Z},\ m \in [1,2)$$

$$\log_2(x_{\log}) = k + \log_2(m)$$



Content

TQA approximation For 2^x and $\log_2(1+x),\ x \in [0,1)$ (1)

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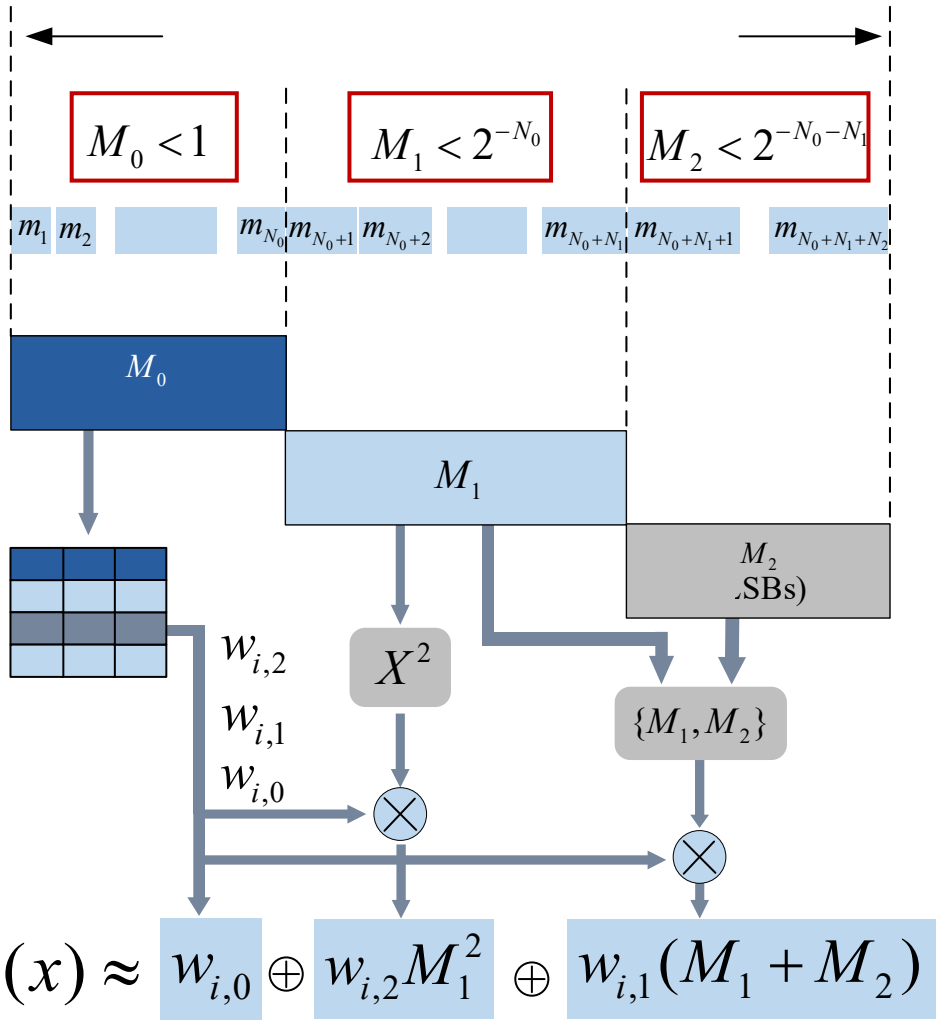
Quadratic Approximation and Fine-Grained Part



To meet the demand for **high-precision** computation, quadratic polynomial approximation was used.

- Input is split into **three** parts:

$$x = \{M_0, M_1, M_2\}$$

$$\begin{cases} M_0 < 1 \\ M_1 < 2^{-N_0} \\ M_2 < 2^{-N_0-N_1} \end{cases}$$

- Uniform Segments:

M_0 for fitted coefficients $w_{i,2}$

- only M_1 for quadratic term :

$w_{i,2} \cdot M_1^2$ (to avoid wasting hardware resouce on LSBs)

- (M_1+M_2) for linear term :

$w_{i,1} \cdot (M_1+M_2)$

TQA approximation For 2^x and $\log_2(1+x),\ x \in [0,1)$ (2)

Error analysis is necessary for coefficient quantization, truncation error, and curve fitting.

Variable	Bit-width
$w_{i,2}$	N_{w2}
$w_{i,1}$	N_{w1}
$w_{i,0}$	N_{w0}
M_0	N_0
M_1	N_1
M_1+M_2	N_{12}
M_1^2	N_{a1}
$w_{i,2}M_1^2$	N_{a2}
$w_{i,1}(M_1+M_2)$	N_{a3}

Overall Approximation Error (*Error*)

Coefficient Quantization Error: (e_c)

$$w_{i,2} \dashrightarrow w_{i,2}+e_{c,2}$$

$$w_{i,1} \dashrightarrow w_{i,1}+e_{c,1}$$

$$w_{i,0} \dashrightarrow w_{i,0}+e_{c,0}$$

Truncation Error: (e_t)

$$(M_1+M_2)^2 \dashrightarrow M_1^2+e_{t1}$$

$$(e_{t1}=2M_1M_2+M_2^2 \approx 2M_1M_2)$$

$$(M_1+M_2) \dashrightarrow M_1+M_2+e_{t2}$$

Model Error: (e_m)

$$y_{exact} \dashrightarrow y_{fit}+e_m$$

(More segments, Smaller model error.)

TQA approximation For 2^x and $\log_2(1+x),\ x \in [0,1)$ (3)

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Second-order polynomial expression:

$$y(x) \approx w_{i,2}M_1^2 + w_{i,1}(M_1 + M_2) + w_{i,0}$$

$$w_{i,2} \dashrightarrow w_{i,2} + e_{c,2}$$
$$w_{i,1} \dashrightarrow w_{i,1} + e_{c,1}$$
$$w_{i,0} \dashrightarrow w_{i,0} + e_{c,0}$$
$$(M_1 + M_2)^2 \dashrightarrow M_1^2 + e_{t1}$$
$$(M_1 + M_2) \dashrightarrow M_1 + M_2 + e_{t2}$$

Expression with error:

$$y(x) \approx (w_{i,2} + e_{c,2})(M_1^2 + e_{t1}) + (w_{i,1} + e_{c,1})(M_1 + M_2 + e_{t2}) + (w_{i,0} + e_{c,0})$$

subtraction

$$Error = 2w_{i,2}M_1M_2 + w_{i,1}e_{t2} + M_1^2e_{c,2} + (M_1 + M_2)e_{c,1} + e_{c,0} + e_{c,2}e_{t1} + e_{c,1}e_{t2}$$

small enough to ignore them

$$Error \approx 2w_{i,2}M_1M_2 + w_{i,1}e_{t2} + M_1^2e_{c,2} + (M_1 + M_2)e_{c,1} + e_{c,0}$$

$$Error < w_{i,2} \times 2^{-(2N_0+N_1+1)} + w_{i,1} \times 2^{-(N_0+N_{12})} + 2^{-(2N_0+Nw_2)} + 2^{-(N_0+Nw_1)} + 2^{-(Nw_0)}$$

$a+b<c$

$|a|+|b|<c$

$$Error < |w_{i,2}| \times 2^{-(2N_0+N_1+1)} + |w_{i,1}| \times 2^{-(N_0+N_{12})} + 2^{-(2N_0+Nw_2)} + 2^{-(N_0+Nw_1)} + 2^{-(Nw_0)} \le 2^{-N_{ulp}}$$

TQA approximation For 2^x and $\log_2(1+x),\ x \in [0,1)$ (4)

$$|w_{i,2}|\times 2^{-(2N_0+N_1+1)} + |w_{i,1}|\times 2^{-(N_0+N_{12})} + 2^{-(2N_0+Nw_2)} + 2^{-(N_0+Nw_1)} + 2^{-(Nw_0)} \le 2^{-N_{ulp}}$$

allocate $2^{-N_{ulp}}$ among these five terms

$$|w_{i,2}|\times 2^{-(2N_0+N_1+1)} \le \frac{1}{5}\times 2^{-N_{ulp}} \Rightarrow N_1 \ge N_{ulp} + \log_2(5) + \log_2(|w_{i,2}|) - 2N_0 + 1$$

$$|w_{i,1}|\times 2^{-(N_0+N_{12})} \le \frac{1}{5}\times 2^{-N_{ulp}} \Rightarrow N_{12} \ge N_{ulp} + \log_2(5) + \log_2(|w_{i,1}|) - N_0$$

$$2^{-(2N_0+Nw_2)} \le \frac{1}{5}\times 2^{-N_{ulp}} \Rightarrow N_{w2} \ge N_{ulp} + \log_2(5) - 2N_0$$

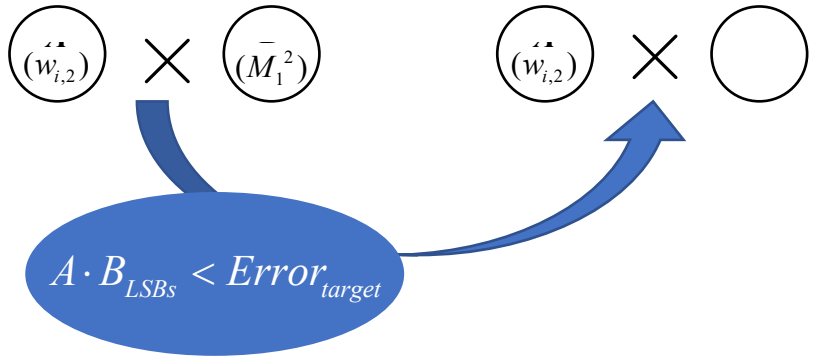
$$2^{-(N_0+Nw_1)} \le \frac{1}{5}\times 2^{-N_{ulp}} \Rightarrow N_{w1} \ge N_{ulp} + \log_2(5) - N_0$$

$$2^{-(Nw_0)} \le \frac{1}{5}\times 2^{-N_{ulp}} \Rightarrow N_{w0} \ge N_{ulp} + \log_2(5)$$

Variable	Bit-width
$w_{i,2}$	N_{w2}
$w_{i,1}$	N_{w1}
$w_{i,0}$	N_{w0}
M_0	N_0
M_1	N_1
$M_1 + M_2$	N_{12}
M_1^2	N_{a1}
$w_{i,2}M_1^2$	N_{a2}
$w_{i,1}(M_1+M_2)$	N_{a3}

TQA approximation For 2^x and $\log_2(1+x),\ x \in [0,1)$ (5)

The bit-width of $M_1^{\,2}$ (N_{a1}) can be decided through below.



$$w_{i,2}M_1^{\,2} = w_{i,2}\left[(M_1^{\,2})_{LSBs} + (M_1^{\,2})_{MSBs}\right] \approx w_{i,2} \cdot (M_1^{\,2})_{LSBs}$$

$$w_{i,2}M_1^{\,2} \rightarrow w_{i,2}M_1^{\,2} + e_{a1}$$

$$e_{a1} = w_{i,2} \times 2^{-N_{a1}} < 2^{-N_{ulp}} \rightarrow N_{a1} > N_{ulp} + \log_2(\max(w_{i,2}))$$

The bit-width of $w_{i,2}M_1^{\,2}$, $w_{i,1}(M_1+M_2)$, $w_{i,0}$ (N_{a2}, N_{a3}, N_{w0}) can be decided through below.

$$y(x) \approx [w_{i,2}M_1^2 + e_{a2}] + [w_{i,1}(M_1+M_2) + e_{a3}] + [w_{i,0} + e_{a4}]$$

allocate $2^{-N_{ulp}}$

$$e_{a2} < 2^{-N_{a2}} \le \tfrac{1}{3} \times 2^{-N_{ulp}} \rightarrow N_{a2} \ge \log_2(3) + N_{ulp}$$

$$e_{a3} < 2^{-N_{a3}} \le \tfrac{1}{3} \times 2^{-N_{ulp}} \rightarrow N_{a3} \ge \log_2(3) + N_{ulp}$$

$$e_{a4} < 2^{-N_{a4}} \le \tfrac{1}{3} \times 2^{-N_{ulp}} \rightarrow N_{w0} \ge \log_2(3) + N_{ulp}$$

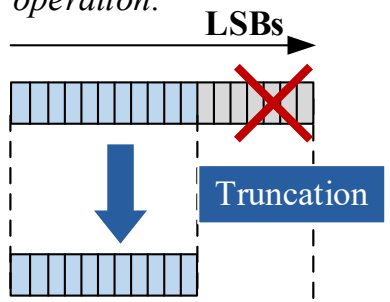
Variable	Bit-width
M_1^2	N_{a1}
$w_{i,2}M_1^2$	N_{a2}
$w_{i,1}(M_1+M_2)$	N_{a3}

Hardware-Oriented Algorithmic Optimization

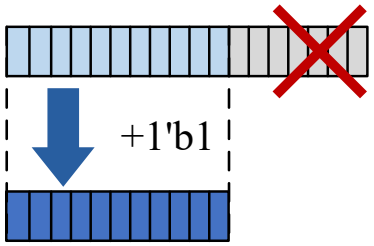
Coefficient Quantization

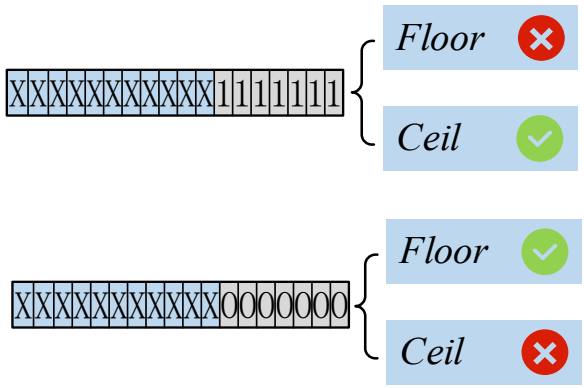
$$X\_floor = \frac{floor(X \times 2^N)}{2^N} \qquad X\_ceil = \frac{ceil(X \times 2^N)}{2^N}$$

Floor operation:



Ceil operation:

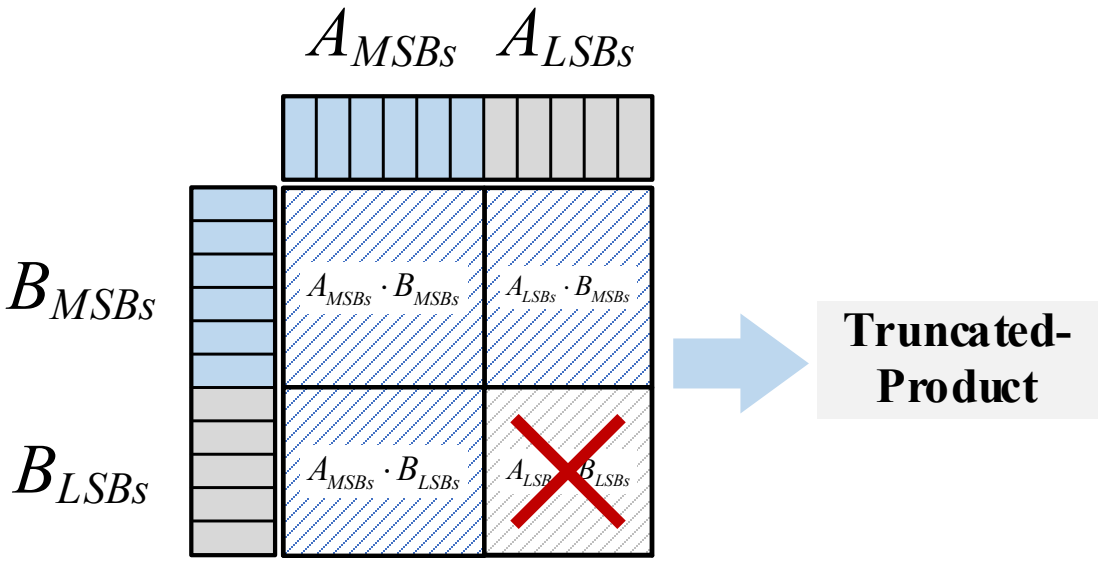




$[w_{i,2}, w_{i,1}, w_{i,0}]$,
which make error minimal

Truncated Multiplier

$$A = A_{MSBs} + A_{LSBs} \qquad B = B_{MSBs} + B_{LSBs}$$



$$Truncated\ \ product(A \times B)$$
$$\approx A_{MSBs} \times B_{MSBs} + A_{MSBs} \times B_{LSBs} + A_{LSBs} \times B_{MSBs}$$

Content

The High Precision Softmax Architecture

$$f_2(x_i)=\frac{2^{x_i}}{\sum_{j=1}^{N}2^{x_j}}=\frac{2^{x_i-x_{max}}}{\sum_{j=1}^{N}2^{x_j-x_{max}}}=2^{x_i-x_{max}-\log_2(\sum_{j=1}^{N}2^{x_j-x_{max}})}$$

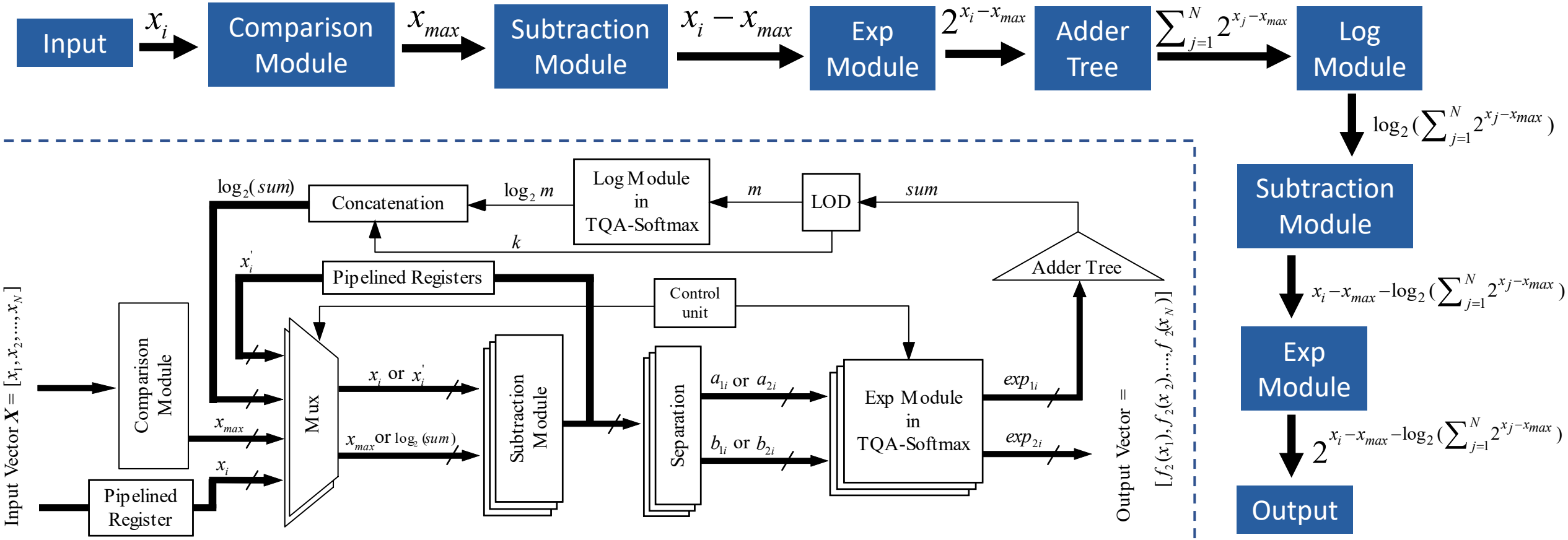
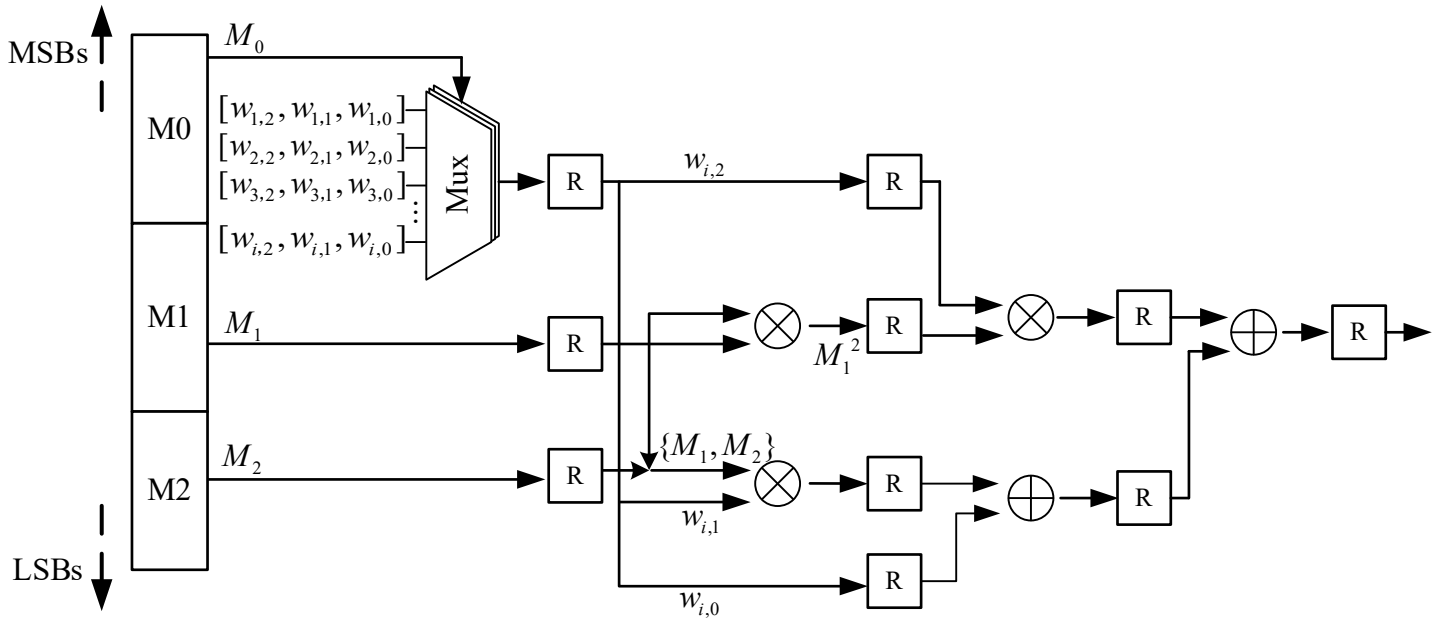


Fig.2 :TQA-Softmax Architecture

TQA Core Compute Unit



- M_0 for $w_{i,2}, w_{i,1}, w_{i,0}$.
- M_1 for quadratic term.
- M_1 & M_2 for linear term.

Content

Experiment Results (1)

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More Segments, Smaller Model Error （error from curve fitting）

$$Error_{target} \leq 1 \times 10^{-6}$$

$$N_{ulp} \geq -log_2(Error_{target}) \approx 19.93 \Rightarrow N_{ulp} = 21$$

Table I : Model Error of Exponent and Logarithm

N0	Exponent		Logarithm	
	MACE	MAE	MACE	MAE
3	1.04e-5	2.54e-6	3.96e-5	5.74e-6
4	**1.33e-6**	**3.17e-7**	5.42e-6	7.15e-7
5	1.68e-7	3.97e-8	**7.05e-7**	**8.94e-8**

(MAE: Mean Absolute Error)
(MACE: Maximum Absolute Calculation Error)

Table II: Numerical Error Analysis

Method	Input range	MACE	MAE
TQA-Acc	[-1,1]	8.34e-7	2.15e-7
TQA-App		1.04e-6	2.26e-7
TQA-Acc	[-5,5]	2.08e-6	2.30e-7
TQA-App		2.50e-6	2.60e-7
TQA-Acc	[-10,10]	2.92e-6	2.52e-7
TQA-App		2.98e-6	2.75e-7
[12]	—	3.81e-6	—

(TQA-Acc: using accurate multipliers)
(TQA-App: using truncated multipliers)

[12] Y. Zhang, L. Peng, L. Quan, Y. Zhang, S. Zheng, and H. Chen, “High precision method and architecture for base-2 softmax function in DNN training,” IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 70, no. 8, pp. 3268–3279, 2023.

Experiment Results (2)

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Table III: Comparison of Hardware Synthesis Results

[1]Design	Area (μm^2)	Power (mW)	[2]Latency	[3]ADP ($mm^2 \cdot ns$)	[4]Efficiency (G/s/mm^2 ·mW)
TQA-Acc	26028.20	28.60	18	0.468	10.75
TQA-App	24891.80	27.44	18	0.448	11.71
CORDIC-based[12]	98787.43	24.27	26	2.568	3.276
PWL-based[30]	166804.48	40.70	17	2.836	1.178
LUT-based[14]	482304.12	45.95	17	8.199	0.361

[1]Design: All designs are synthesized at TSMC 28nm with a 1GHz clock frequency and a throughput of 8G/s.

[2]Latency: Computation Latency (clock cycles).

[3]ADP: Area-Delay Product ($mm^2 \cdot ns$).

[4]Efficiency = Throughput / (Area · Power), Throughput = Inputs × Frequency.

- Compared with TQA-Acc:

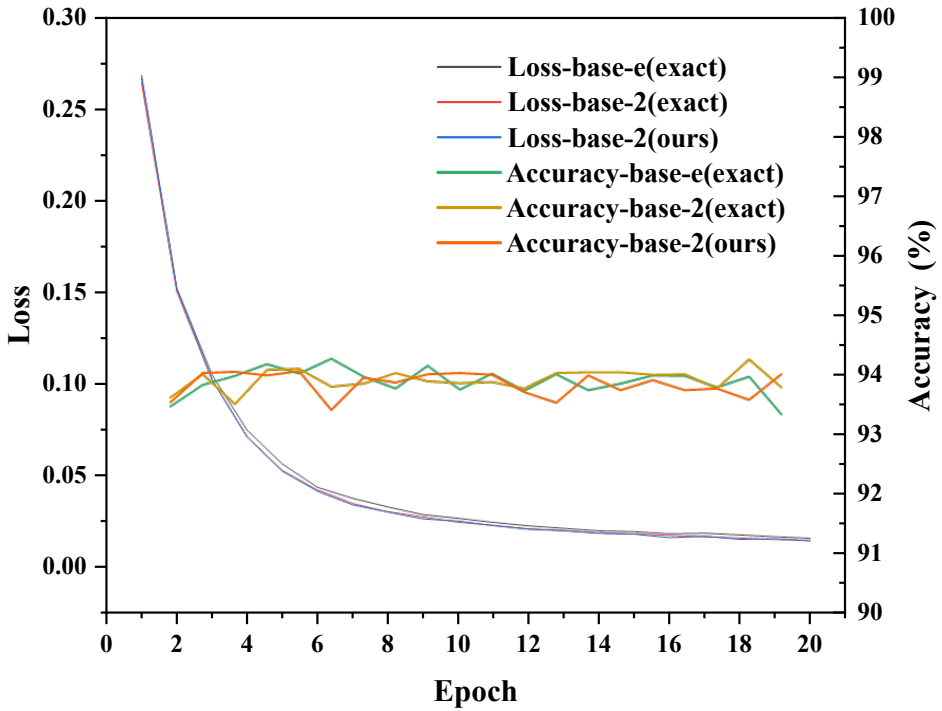
Area	↘	4.37%
Power	↘	4.06%
ADP	↘	4.27%
Efficiency	↗	8.93%

- Compared with [12]:

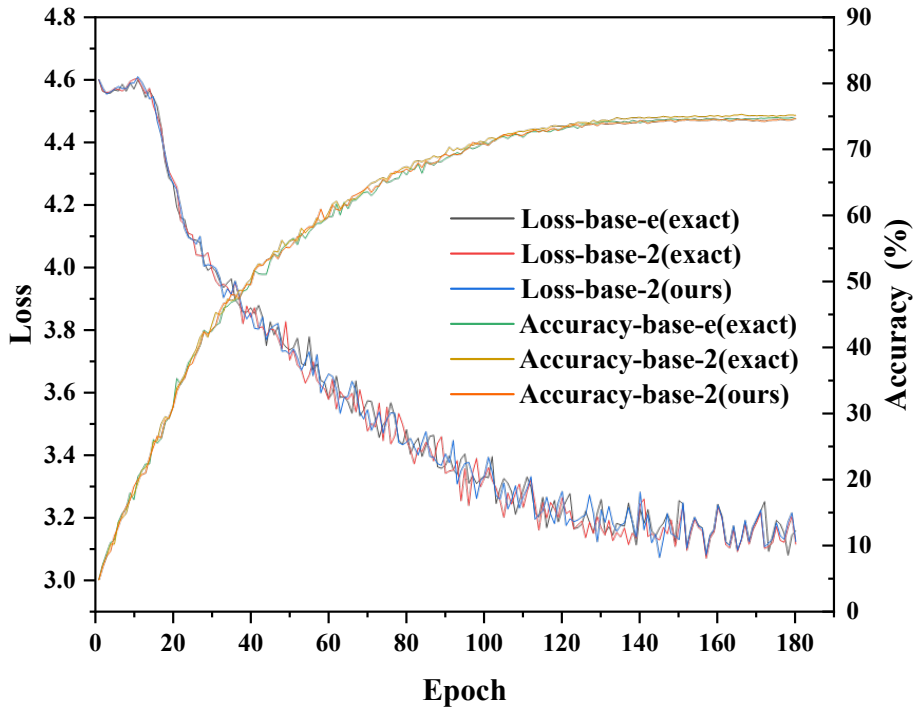
Area	↘	74.80%
Power	↗	+ 3.17
ADP	↘	- 0.448
Efficiency	↗	3.57×

[12] Y. Zhang, L. Peng, L. Quan, Y. Zhang, S. Zheng, and H. Chen, "High precision method and architecture for base-2 softmax function in DNN training," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 70, no. 8, pp. 3268–3279, 2023.
[14] Y. Zhang, Y. Zhang, L. Peng, L. Quan, S. Zheng, Z. Lu, and H. Chen, "Base-2 softmax function: Suitability for training and efficient hardware implementation," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 69, no. 9, pp. 3605–3618, 2022.
F. Lyu, X. Xu, Y. Wang, Y. Luo, Y. Wang, and H. Pan, "Ultralow latency VLSI architecture based on a linear approximation method for computing Nth roots of floating-point numbers," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 68, no. 2, pp. 715–727, 2020.

Experiment Results (3)



(a) ViT (training from scratch)



(b) BERT-based Model (fine-tuning)

- Base-2 softmax can be used in Transformer models.
- The proposed approximation closely matches the exact baselines.

Content

Conclusions

- This paper proposes a high-precision, low-latency base-2 softmax hardware architecture for Transformer training. It achieves a mean absolute error on the order of **10^{-7}** .
- Through systematic error analysis and hardware-oriented algorithmic optimizations, the design achieves a highly robust architecture. Compared to state-of-the-art solutions, it reduces hardware area by up to **74.8%** and improves efficiency by **3.57×** .
- Network-level training experiments on Transformer models confirm that the hardware design achieves stable convergence with negligible performance degradation, proving its suitability for high-performance neural network training tasks.

Thanks!