Approximating Floating-Point Addition Using The Geometric Mean

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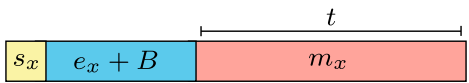
Floating-point system

A normalized FP number can be expressed as

$$x = (-1)^{s_x} 2^{e_x} (1 + m_x),$$

and stored in hardware as



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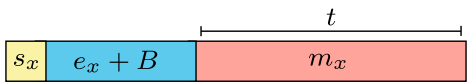
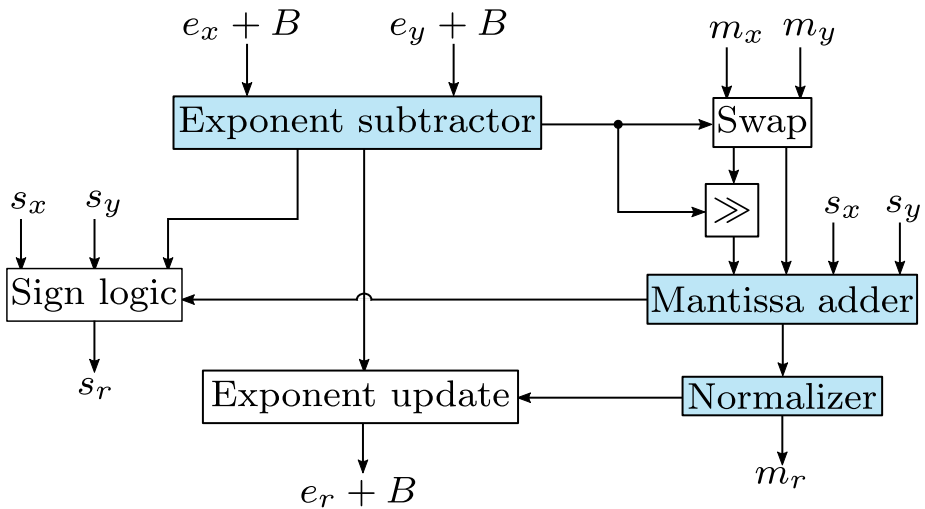


Floating-point addition:

$$\begin{aligned} x \pm |y| &= (-1)^{s_x} 2^{e_x} (1 + m_x) \pm 2^{e_y} (1 + m_y) \\ &= (-1)^{s_x} 2^{e_x} \left(1 + m_x \pm 2^{e_y - e_x} (1 + m_y)\right) \end{aligned}$$

Previous work–Approximate parts in exact designs



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- Fundamentally different approach
  - Based on the geometric mean
  - Integer-only operations

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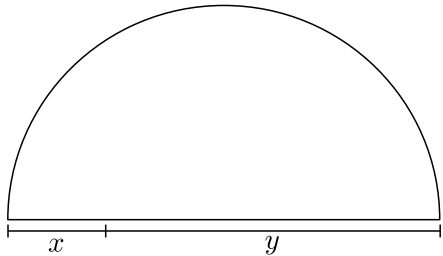
- Fundamentally different approach
  - Based on the geometric mean
  - Integer-only operations
- Suitable for both hardware and software
  - Unique in being applicable for software
  - Usable alongside existing soft FP libraries, e.g. FLIP
- Restricted to positive inputs, still many applications
  - Squares, e.g. vectors norms: $\sqrt{\sum_{j=1} {x_j}^2}$
  - Exponentials, e.g. softmax: $\frac{e^{x_i}}{\sum_{j=1} e^{x_j}}$
  - Certain data, e.g. pixel values

The arithmetic and geometric mean



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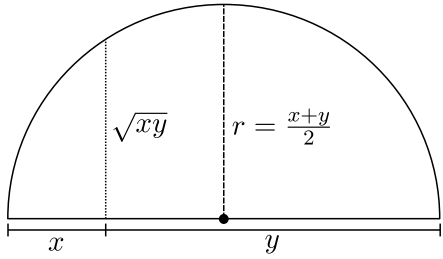

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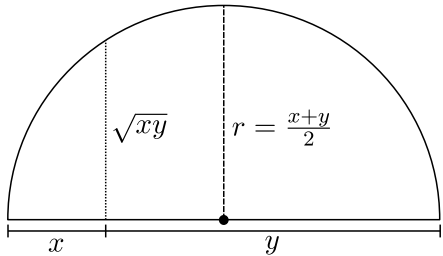
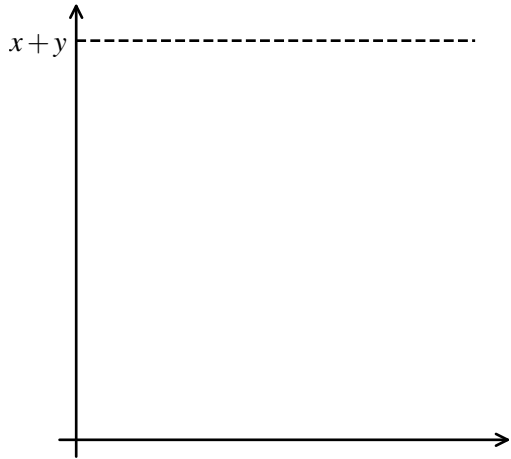


$$x + y = 2r \approx 2\sqrt{xy}$$

Proposed approach



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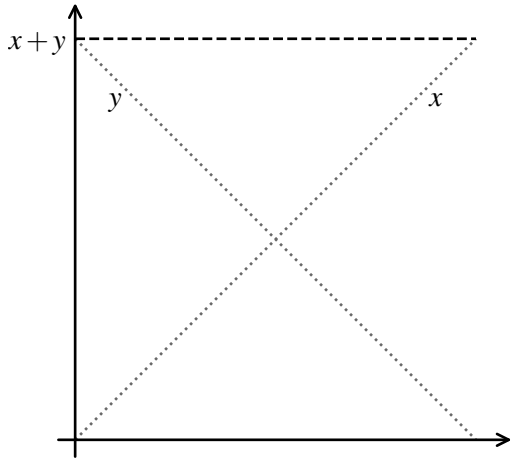

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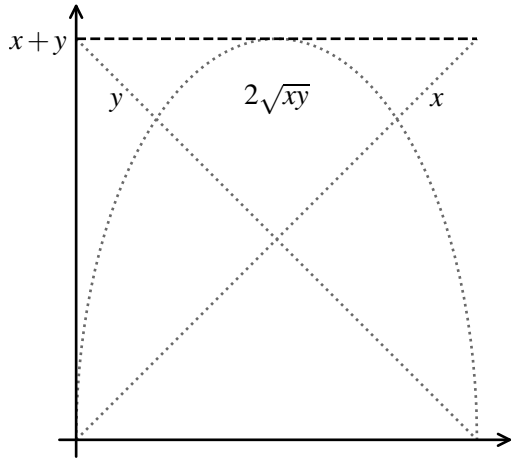
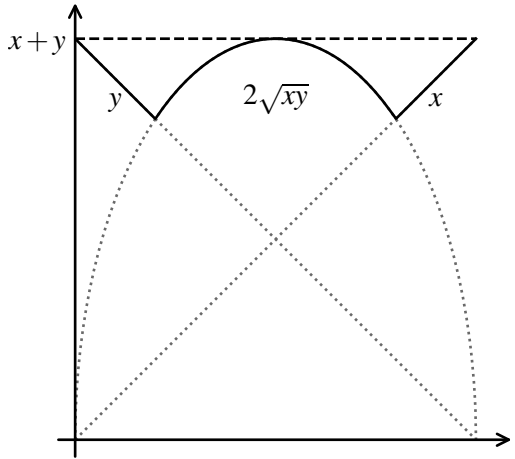


Proposed approach

$$x + y \approx \max\left(x,\, y,\, 2\sqrt{xy}\right)$$



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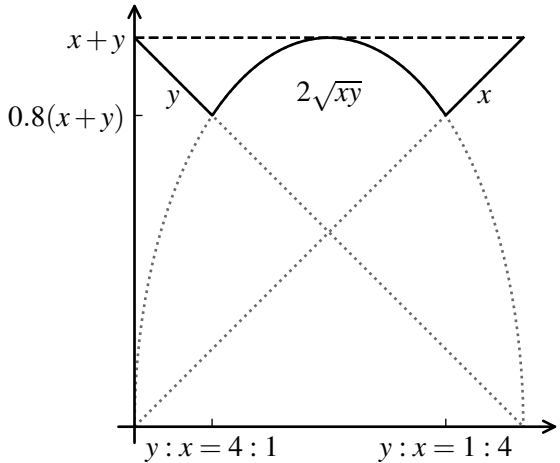

Implementation of $\max\left(x,\, y,\, 2\sqrt{xy}\right)$

1. Approximate to logarithmic fixed-point numbers
2. Perform computations
   - Multiplication $\Rightarrow$ addition, square-root $\Rightarrow$ right shift
3. Approximate back to floating-point

Mitchell's logarithm approximation

Based on

$$\log_2\left(2^e(1+m)\right) = e + \log_2(1+m) \approx e + m,\ 0 \leqslant m < 1$$

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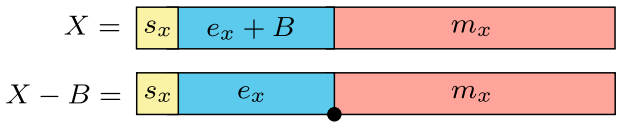
In practice:



$$(-1)^{s_x} \log_2 |x| \approx X - B$$

Implementation in the logarithmic domain

Operation	Mathematical notation	Base expression[1]
Multiplication	$x \times y$	$X + Y - B$
Multiplication by 2	$2x$	$X + (1 \ll t)$
Square root	$\sqrt{x}$	$(X + B) \gg 1$
Maximum	$\max(x,\, y)$	$\max(X,\, Y)$

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This gives

$$\max\left(x,\, y,\, 2\sqrt{xy}\right) \Rightarrow \max\left(X,\, Y,\, (X + Y + C) \gg 1\right)$$

where $C = 2 \ll t$

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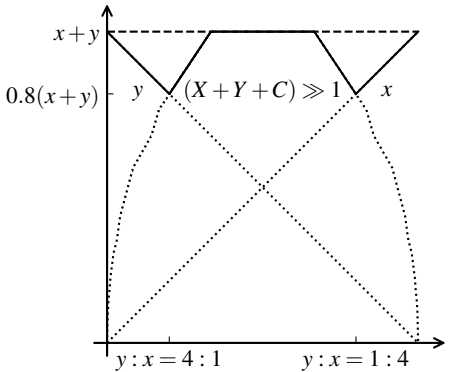
- $C = (2 \ll t) + 1 \Rightarrow$ maximum relative error is exactly -0.2
- Correct rounding when
  - x and y share the same exponent
  - The difference in exponents is greater than t

Analysis of a constant sum



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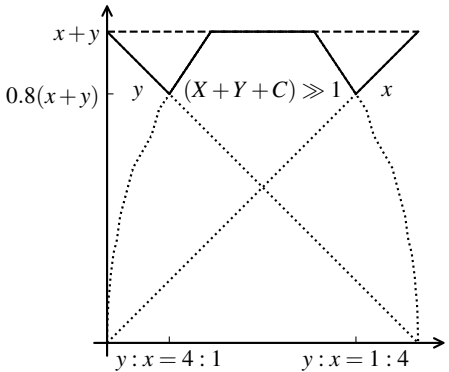

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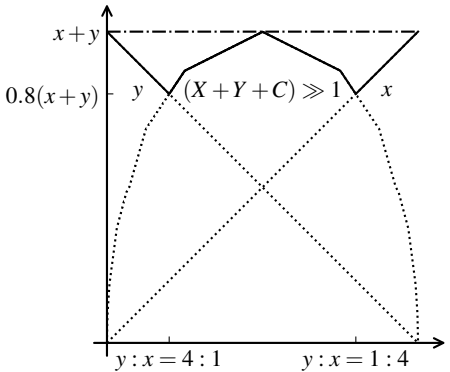
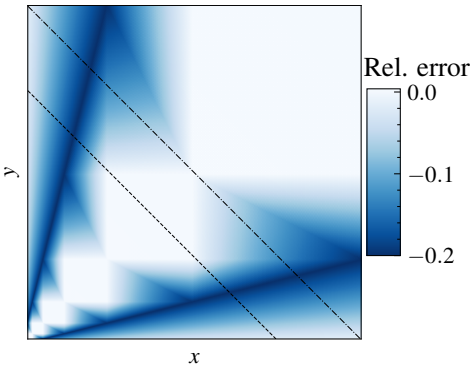


(b) Sum is an exact power of two

Sweep of six binades



(a) Float values

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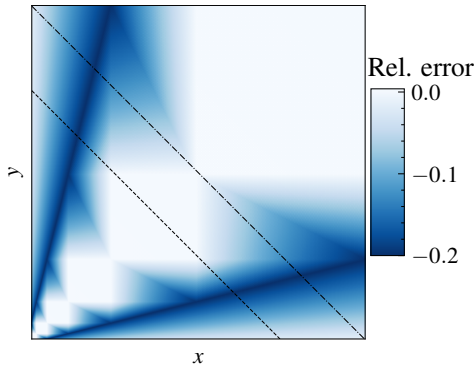

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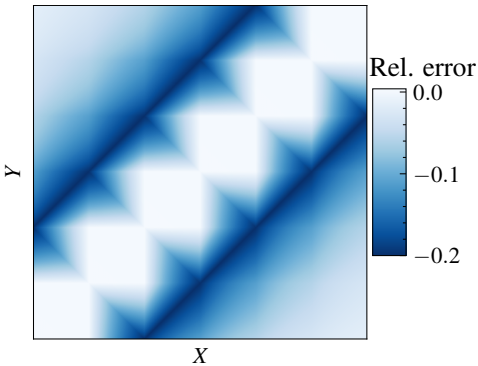
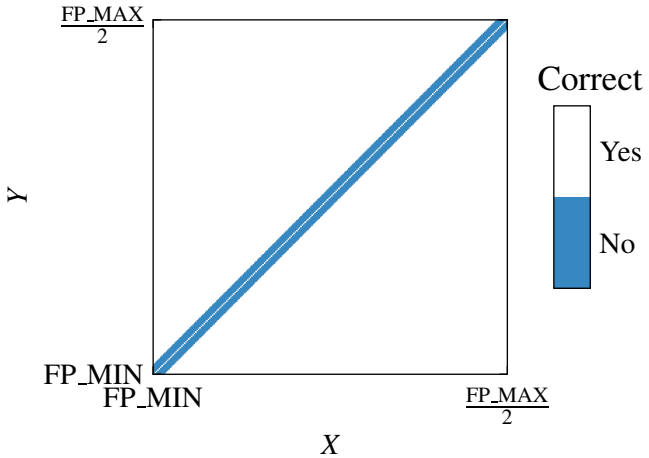


(b) Bit patterns

Correct rounding is often obtained – BF16



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- Subnormal numbers can be
  - Flushed to zero
  - Treated as normalized numbers

Evaluation

- Compared designs
  - Approximate: RLOA I and II[2], ACFPA[3]
  - Exact: FloPoCo

[2] W. Liu Chen, C. Wang, M. O'Neill, and F. Lombardi, "Design and Analysis of Inexact Floating-Point Adders", in *IEEE Trans. Comput., 2016*

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- Five FP format: **E5M2**, E4M3, BF16, FP16, and **FP32**

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Error metrics

Measure	Definition
Relative error distance (RED)	$\left\lvert \frac{\widetilde{x} - x}{x} \right\rvert$
Maximum RED	$\max(\mathrm{RED})$
Mean RED (MRED)	$\mathrm{mean}(\mathrm{RED})$
Normalized MED (NMED)	$\frac{\mathrm{mean}(\lvert\widetilde{x} - x\rvert)}{\texttt{FP\_MAX}}$
Error rate (ER)	Percentage of times not `roundTiesToEven`

Results for E5M2

Design	Max. RED	MRED	NMED	ER
Proposed	$\mathbf{2.00 \times 10^{-1}}$	$\mathbf{2.85 \times 10^{-2}}$	$\mathbf{1.05 \times 10^{-3}}$	**13.2%**
RLOA I	5.00×10^{-1}	4.81×10^{-2}	2.08×10^{-3}	18.7%
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Design	Delay, ns	LUTs
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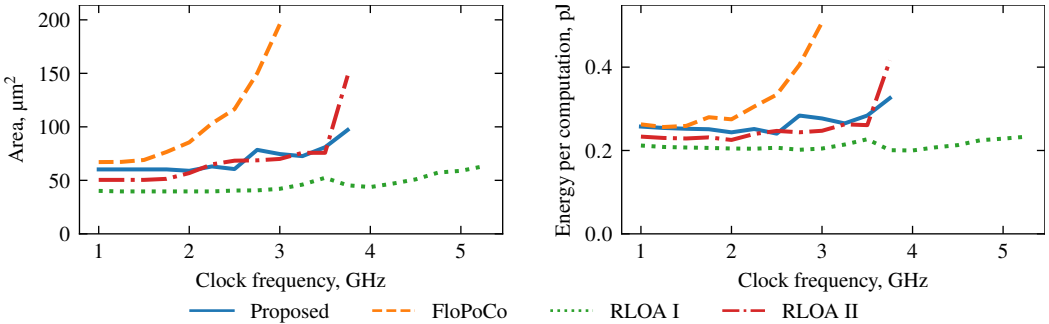

Results for FP32

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ACFPA Acc=0	4.88×10^{-4}	4.95×10^{-6}	1.53×10^{-8}	18.4%
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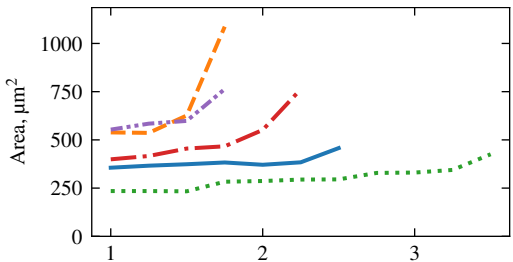

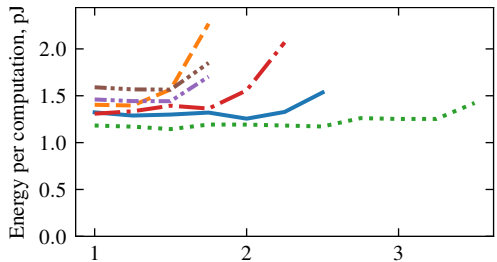

Proposed FloPoCo RLOA I RLOA II ACFPA Acc=0 ACFPA Acc=7

Takeaway: Balances numerical performance and hardware cost

- Compared to designs with less delay on ASIC
  - More accurate
- Compared to more accurate designs
  - Less area on ASIC
  - Shorter delays on both ASIC and FPGA

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- Usable alongside existing soft FP libraries

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$\max(X, Y, (X + Y + C) \gg 1)$ for BF16 on RV64IB:

```
1  addiw   a5, a1, 513     ; a5 = Y + C
2  addw    a5, a5, a0      ; a5 = a5 + X
3  srliw   a5, a5, 1       ; a5 = a5 >> 1
4  maxu    a0, a0, a1      ; a0 = max(X, Y)
5  maxu    a0, a5, a0      ; a0 = max(a5, a0)
```

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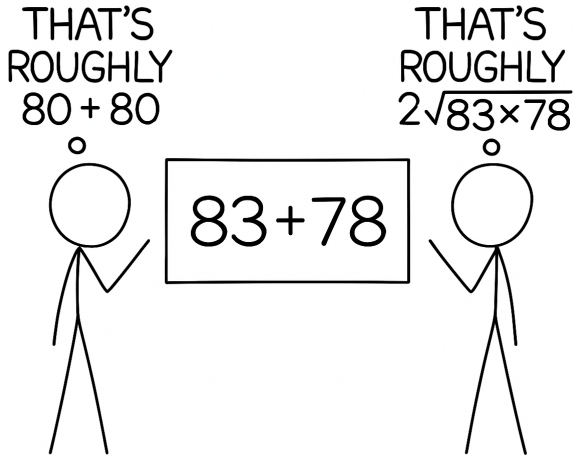
- Derived maximum relative errors
- Favorable trade-off: hardware cost vs. numerical performance
- Attractive for processors


THAT'S ROUGHLY 80 + 80
83+78
THAT'S ROUGHLY 2√83×78

Results for E4M3

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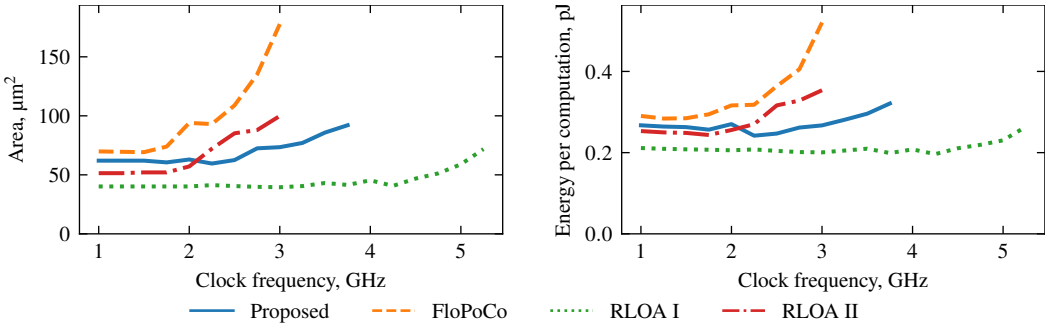
Proposed	$\mathbf{2.00 \times 10^{-1}}$	5.42×10^{-2}	4.96×10^{-3}	43.9%
RLOA I	5.00×10^{-1}	8.98×10^{-2}	9.35×10^{-3}	48.3%
RLOA II	3.60×10^{-1}	$\mathbf{4.39 \times 10^{-2}}$	$\mathbf{4.75 \times 10^{-3}}$	**30.6%**

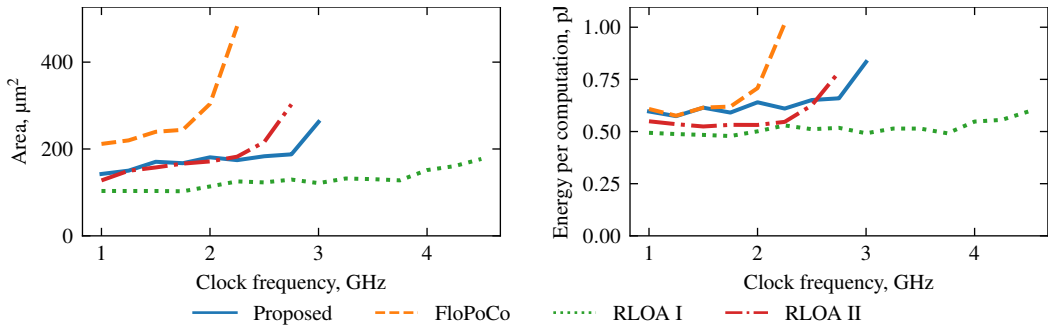
Design	Delay, ns	LUTs
Proposed	4.103	27
RLOA I	**2.981**	**15**
RLOA II	3.967	28
FloPoCo	4.652	37



Results for BF16

Design	Max. RED	MRED	NMED	ER
Proposed	$\mathbf{2.00 \times 10^{-1}}$	3.62×10^{-3}	1.30×10^{-5}	6.3%
RLOA I	5.00×10^{-1}	4.60×10^{-3}	2.18×10^{-5}	5.9%
RLOA II	3.35×10^{-1}	$\mathbf{1.49 \times 10^{-3}}$	$\mathbf{8.28 \times 10^{-6}}$	**3.9%**

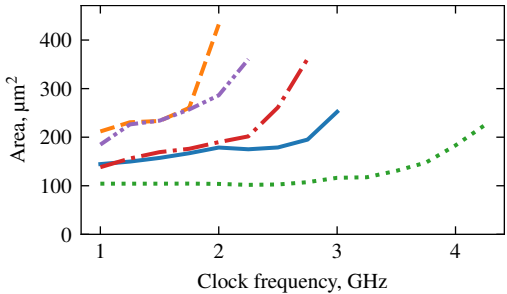
Design	Delay, ns	LUTs
Proposed	4.521	55
RLOA I	**4.396**	**46**
RLOA II	6.556	56
FloPoCo	7.913	75

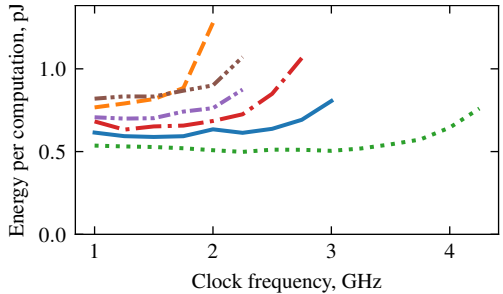


Results for FP16

Design	Max. RED	MRED	NMED	ER
Proposed	2.00×10^{-1}	2.88×10^{-2}	9.91×10^{-4}	61.0%
RLOA I	5.00×10^{-1}	3.78×10^{-2}	1.64×10^{-3}	58.7%
RLOA II	3.34×10^{-1}	1.16×10^{-2}	6.01×10^{-4}	39.6%
ACFPA Acc=0	2.00×10^{-1}	1.49×10^{-2}	4.77×10^{-4}	62.5%
ACFPA Acc=7	$\mathbf{7.75 \times 10^{-3}}$	$\mathbf{5.19 \times 10^{-4}}$	$\mathbf{1.86 \times 10^{-5}}$	**37.8%**

Design	Delay, ns	LUTs
Proposed	4.850	55
RLOA I	**3.829**	**41**
RLOA II	7.086	57
ACFPA	7.010	94
FloPoCo	7.209	79





Proposed FloPoCo RLOA I RLOA II ACFPA Acc=0 ACFPA Acc=7

Why is correct rounding obtained for inputs sharing exponent?

Consider

$$
\begin{aligned}
x + y &= 2^{e_x}(1 + m_x) + 2^{e_x}(1 + m_y) && (1)\\
&= 2^{e_x}(2 + m_x + m_y) && (2)\\
&= 2^{e_x+1}\left(1 + \frac{m_x + m_y}{2}\right) && (3)
\end{aligned}
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\end{aligned}
$$

and

$$
\begin{aligned}
(X + Y + C) \gg 1 \Rightarrow \left(E_x + m_x + E_x + m_y + 2 + 2^{-t}\right) \gg 1 && (4)\\
= E_x + 1 + \left\lfloor \frac{m_x + m_y + 2^{-t}}{2} \right\rfloor && (5)
\end{aligned}
$$